

Name: Solutions  
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Date: 8 June 2015

Do not start this exam until instructed; you will have 90 minutes to finish the exam. No notes, books, calculators, phones or electronic devices are allowed on this exam. If you have a question, raise your hand; otherwise, there is no talking during the exam.

There are 14 problems on this exam on 6 pages, in addition to this cover page. The point values of each problem vary, but are listed in the questions.

Good luck!



From *SMBC*.

**Fill in the Blanks Section.** No work needed, and no partial credit available. [4+4+6+4+4+3=25]

1. (4 points) A vector normal to the plane  $3x + y = 7z$  is  $\vec{n} = \langle 3, 1, -7 \rangle$ .

$$3x + y - 7z = 0$$

$$\langle 3, 1, -7 \rangle \cdot \langle x, y, z \rangle = 0$$

2. (4 points) A surface is given by  $F(x, y, z) = x^3 + 2yz + y^2$ . The equation of the tangent plane at the point  $(1, 0, 1)$  is given by  $\underline{3(x-1) + 2y = 0}$ .

$$0 = F_x(1, 0, 1)(x-1) + F_y(1, 0, 1)(y-0) + F_z(1, 0, 1)(z-1)$$

$$F_x = 3x^2 \quad F_y = 2z + 2y \quad F_z = 2y$$

3. (2+2+2=6 points) A particle has acceleration  $\vec{a}$ , velocity  $\vec{v}$  and position  $\vec{r}$ . You are given that

$$\vec{a}(t) = \vec{i} - 3\vec{j}$$

$$\vec{v}(0) = \vec{k}$$

$$\vec{r}(0) = 2\vec{j} + \vec{k}$$

Find the following:

$$(a) \vec{v}(t) = \underline{\vec{i}t - 3t\vec{j} + \vec{k}}$$

$$(b) \vec{r}(t) = \underline{\frac{1}{2}t^2\vec{i} + (2 - \frac{3}{2}t^2)\vec{j} + t\vec{k}}$$

(c) Does the particle go through the origin? No

Extra Work Space.

$$\begin{aligned} \vec{v}(t) &= \underbrace{\vec{v}(0)}_{\vec{k}} + \int_0^t \vec{a} = \vec{i} - 3\vec{j} \, ds \\ &= t\vec{i} - 3t\vec{j} + \vec{k} \\ \vec{r}(t) &= \vec{r}(0) + \int_0^t \vec{v} = \vec{j} - 3\vec{s}\vec{j} + \vec{k} \, ds \\ &= \frac{1}{2}t^2\vec{i} + (2 - \frac{3}{2}t^2)\vec{j} + t\vec{k} \end{aligned}$$

$$\text{Set } \vec{r}(t) = \vec{0}:$$

$$\left. \begin{array}{l} \frac{1}{2}t^2 = 0 \\ 2 - \frac{3}{2}t^2 = 0 \\ t = 0 \end{array} \right\} \text{No solution.}$$

4. (4 points) Suppose that  $(3, 4)$  is a critical point for the surface  $h(x, y)$ , and say that

$$h_{xx}(3, 4) = 6, \quad h_{yy} = 1, \quad h_{xy}(3, 4) = -2$$

Choose one of the following:

- (a)  $(3, 4)$  is a local maximum of  $h$ .
- (b)  $(3, 4)$  is a local minimum of  $h$ .
- (c)  $(3, 4)$  is a saddle point of  $h$ .
- (d) There is not enough information to determine this.

$$\begin{aligned} D = h_{xx} h_{yy} - (h_{xy})^2 &= 2 > 0 \\ h_{xx} &> 0 \end{aligned} \quad \Rightarrow \quad \text{Min.}$$

5. (4 points) The two legs of a right triangle are measured to be 2 cm and 4 cm with a possible error of at most 0.3 cm in each. Use differentials to estimate the maximum error in the calculated value of the area of the triangle: 0.9 cm<sup>2</sup>.

$$A = \frac{1}{2}xy$$

$$dA = \frac{1}{2}y \, dx + \frac{1}{2}x \, dy = \frac{1}{2} \cdot 4 \cdot .3 + \frac{1}{2} \cdot 2 \cdot .3 = .9$$

6. (3 points) Consider  $\vec{a} = \langle 1, 0, 0 \rangle$  and  $\vec{b} = \langle 4, 5, -1 \rangle$ . Then the vector projection  $\text{proj}_{\vec{a}} \vec{b}$  is  $\langle 4, 0, 0 \rangle$ .

Extra Work Space.

$$\begin{aligned} \text{proj}_{\vec{a}} \vec{b} &= \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} \frac{\vec{a}}{|\vec{a}|} \\ &= \frac{4+0+0}{1} \frac{\vec{a}}{1} = 4\vec{a}. \end{aligned}$$

**Standard Response Questions.** Show all work to receive credit. [10 + 15 + 10 + 5 + 10 + 10 + 10 + 5 = 75]

7. (10 points) Find a value of  $a$  such that  $u(x, t) = \sin(4t) \cos(ax)$  satisfies the differential equation  $u_{tt} = 4u_{xx}$ .

$$u_t = 4 \cos 4t - a \sin ax$$

$$u_{tt} = -16 \sin 4t - a^2 \sin ax = -16u.$$

$$u_x = -a \sin 4t - a \cos ax$$

$$u_{xx} = -a^2 \sin 4t - a^2 \cos ax = -a^2 u.$$

$$u_{tt} = 4u_{xx} \Rightarrow -16u = -4a^2 u \Rightarrow a^2 = 4$$

$$\boxed{a = \pm 2 \text{ works}}$$

8. (5+10=15 points) Evaluate the following limits. If one or both does not exist, say so.

(a)

$$\lim_{(x,y) \rightarrow (-1,3)} \frac{2xy}{x^2 + y^2}$$

Rational function  $\neq$  point in domain

$\therefore$  Limit = value.

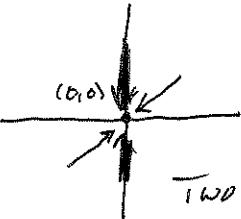
$$\frac{2(-1)(3)}{(-1)^2 + (3)^2} = -\frac{6}{10} = \boxed{-\frac{3}{5}}$$

(b)

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2xy}{x^2 + y^2}$$

Approach when  $x = 0$ :  $\lim_{y \rightarrow 0} \frac{0}{0^2 + y^2} = 0$ .

Approach when  $x = y$ :  $\lim_{y \rightarrow 0} \frac{2y^2}{y^2 + y^2} = \lim_{y \rightarrow 0} 1 = 1$



TWO paths.

Not same.

$$\boxed{\text{Limit Does Not Exist}}$$

9. (7+3=10 points) Consider  $\vec{r}(t) = \langle 2 \sin t, t, 2 \cos t \rangle$ .

(a) Find the arc length function for  $\vec{r}(t)$  starting from the point  $(0, 0, 2)$ .

$$s(t) = \int_0^t |\vec{r}'(u)| du \quad \xrightarrow{\text{This is } t=0.}$$

$$\vec{r}'(t) = \langle 2 \cos t, 1, -2 \sin t \rangle \rightarrow |\vec{r}'(t)| = \sqrt{1 + 4 \cos^2 t + 4 \sin^2 t} = \sqrt{5}.$$

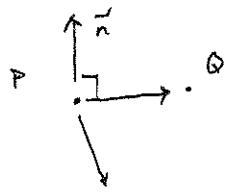
$$\therefore s(t) = \int_0^t \sqrt{5} du = \boxed{\sqrt{5}t}$$

(b) Suppose you move 1 unit along  $\vec{r}(t)$  in the positive direction. Where are you now?

$$s(t) = 1 \Rightarrow t = \frac{1}{\sqrt{5}}.$$

$$\boxed{\vec{r}\left(\frac{1}{\sqrt{5}}\right) = \left\langle 2 \sin \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 2 \cos \frac{1}{\sqrt{5}} \right\rangle}$$

10. (5 points) Find a vector normal to the plane passing through the points  $P = (1, 2, 3)$ ,  $Q = (1, 0, 0)$ , and  $R = (2, 2, 2)$ .



$$\vec{P} - \vec{Q} = \langle 0, 2, 3 \rangle$$

$$\vec{P} - \vec{R} = \langle -1, 0, 1 \rangle$$

Want vector  $\perp$  to both  $\rightarrow$  cross product.

$$\begin{aligned} \vec{n} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 2 & 3 \\ -1 & 0 & 1 \end{vmatrix} = \vec{i} \begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} - \vec{j} \begin{vmatrix} 0 & 3 \\ -1 & 1 \end{vmatrix} + \vec{k} \begin{vmatrix} 0 & 2 \\ -1 & 0 \end{vmatrix} \\ &= \vec{i} (2 - 0) - \vec{j} (0 + 3) + \vec{k} (0 + 2) \end{aligned}$$

$$\boxed{\vec{n} = 2\vec{i} - 3\vec{j} + 2\vec{k}}$$

11. (10 points) Find the linearization of  $f(x, y) = 2 + \sqrt{1+x+\sin y}$  at the point  $(0, \pi)$ .

$$f_x(0, \pi) = \frac{1}{2\sqrt{1+x+\sin y}} \Big|_{(0, \pi)} = \frac{1}{2\sqrt{1+0+0}} = \frac{1}{2}.$$

$$f_y(0, \pi) = \frac{1}{2\sqrt{1+x+\sin y}} \cdot \cos y \Big|_{(0, \pi)} = \frac{1}{2}.$$

$$L(x, y) = f(0, \pi) + f_x(0, \pi)(x - 0) + f_y(0, \pi)(y - \pi)$$

$$\boxed{L(x, y) = 3 + \frac{1}{2}x - \frac{1}{2}(y - \pi)}$$

12. (10 points) Find the partial derivative  $\frac{\partial T}{\partial r}$  for

$$T = \frac{v}{u}, \quad u = \frac{2rq^2}{s^2}, \quad v = rs$$

Your final answer should include only the variables  $q, r, s$ .

$$\begin{aligned} \frac{\partial T}{\partial r} &= \frac{\partial T}{\partial u} \frac{\partial u}{\partial r} + \frac{\partial T}{\partial v} \frac{\partial v}{\partial r} \\ &= \left(-\frac{v}{u^2}\right) \left(\frac{2q^2}{s^2}\right) + \left(\frac{1}{u}\right)s \\ &= -\frac{rs}{\frac{4r^2q^4}{s^4}} \cdot \frac{2q^2}{s^2} + \frac{s^2}{2rq^2}s \\ &= -\frac{s^3}{2rq^2} + \frac{s^3}{2rq^2} = \boxed{0} \end{aligned}$$

Check:  $T = \frac{s^3}{2rq^2}$  is independent of  $r$ . —

13. (5+5=10 points) Consider the function  $g(x, y, z) = x + \ln(yz)$ .  $= x + \ln y + \ln z$

(a) Find  $\nabla g$  at the point  $(3, 1, 2)$ .

$$\begin{aligned}\nabla g(3, 1, 2) &= \langle g_x(3, 1, 2), g_y(3, 1, 2), g_z(3, 1, 2) \rangle \\ &= \langle 1, \frac{1}{y} \Big|_{(3,1,2)}, \frac{1}{z} \Big|_{(3,1,2)} \rangle \\ (3, 1, 2) &= \boxed{\langle 1, 1, \frac{1}{2} \rangle}\end{aligned}$$

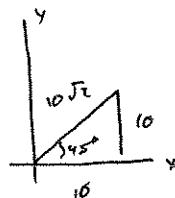
(b) Find the directional derivative of  $g$  at  $\underline{(2, 1, 2)}$  in the direction of  $\vec{i} + \vec{k}$ .

$$\begin{aligned}\text{Unit vector in that direction: } |\vec{i} + \vec{k}| &= \sqrt{1+1} = \sqrt{2} \\ \rightarrow u &= \frac{1}{\sqrt{2}} \vec{i} + \frac{1}{\sqrt{2}} \vec{k}.\end{aligned}$$

$$\begin{aligned}D_u \underline{(3, 1, 2)} &= \nabla g(3, 1, 2) \cdot u \\ &= \langle 1, 1, \frac{1}{2} \rangle \cdot \left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle = \boxed{\frac{3}{2\sqrt{2}}}\end{aligned}$$

14. (5 points) A ball is thrown in the air at an angle of  $45^\circ$  and an initial speed of  $10\sqrt{2}$  m/s. How far away does the ball hit the ground? (Ignore air resistance, and use the value of  $g \approx 10$  m/s<sup>2</sup>).

Velocity:



$$\therefore \vec{v}(0) = \langle 10, 10 \rangle.$$

$$\vec{z}(t) = \langle 0, -10 \rangle.$$

$$\begin{aligned}\Rightarrow \vec{r}(t) &= \langle 10, 10 \rangle + \int_0^t \langle 0, -10 \rangle ds \\ &= \langle 10, 10 - 10t \rangle\end{aligned}$$

$$\begin{aligned}\Rightarrow \vec{r}(t) &= \int_0^t \langle 10, 10 - 10s \rangle ds \\ (\text{set origin = starting point}) &= \langle 10t, 10t - 5t^2 \rangle\end{aligned}$$

$$\text{Set } y \text{ component to 0: } 10t - 5t^2 = 0$$

$$\begin{aligned}x\text{-component at } t=2: & \quad \boxed{20 \text{ m}} \quad \rightarrow t = 2, \underline{2}\end{aligned}$$